High-Order Implicit-Explicit ADER-RK methods for Hyperbolic Systems with Stiff Source Terms in the FLASH Code

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Abstract

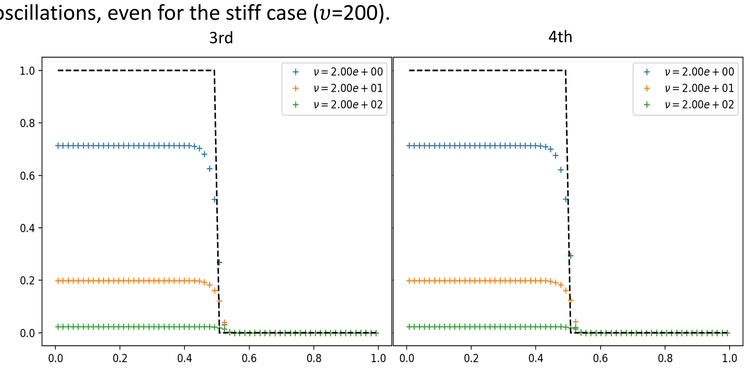
We present the implementation of a high-order in space and time shock-capturing scheme for hyperbolic equations with stiff source terms. The method is based on the Continuous Galerkin formulation of the Arbitrary DERivatives in space and time (ADER-CG[5]/DG[2]) method, but makes use of ordinary differential equation (ODE) integrators to locally advance the solution in time. The method attains high-order accuracy by coupling the local multidimensional spatial information to the local temporal evolution using a Galerkin projection on the governing partial differential equations (PDEs) to reduce the PDE system to a local in space system of ODEs. By making use of mixed implicit-explicit (IMEX) Runge-Kutta (RK) integrators[1] the method may be adapted to stably evolve PDE systems with stiff source terms with less restrictive time steps. The method is benchmarked on stiff linear advection-reaction equations as well as on nonlinear hydrodynamics problems with the FLASH code. This addition promises a significant enhancement to the code's fidelity in modeling laser-driven plasma experiments, particularly when modeling systems with stiff-source terms such as burning or chemical reactions that might occur on time scales much smaller than those given by the hydrodynamics.

Linear Advection with Stiff Source

$$u_t + u_x = S(u)$$

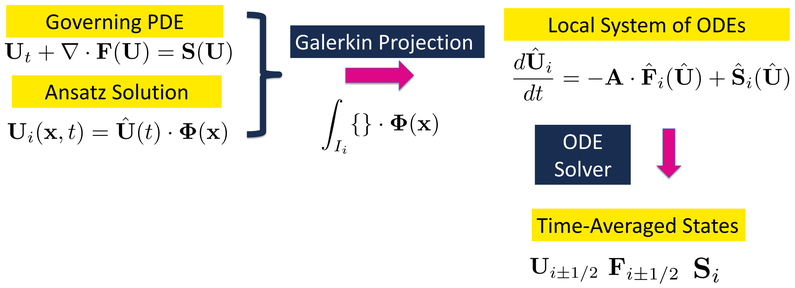
As a first test we consider linear scalar advection with a stiff source term given by $S(u) = -\upsilon u^2$. The problem is given by two time-scales, from advection $a/\Delta x$, and $1/\upsilon$from the source. Solutions to this model PDE will advect the initial profile with speed a=1, while the source term will tend the solution towards the stable equilibrium given by u=0.

For large values of nu>dx the time step for stability can become prohibitively small and the system is said to be stiff. Explicit methods advanced using the advective time step become unstable. We show numerical solutions for a discontinuous initial profile, located at x=0.3 at t=0, advanced up to time T=0.2 using the ADER-RK method with a second order IMEX RK2[1] method for the local space-time predictor. The solution is computed on a grid of 64 cells, using the advective time step with a CFL of 0.75. Solutions are given for various values of υ (colored +'s) along with the analytic solution for nu=0 (dashed line). All solutions are stable without any unphysical oscillations, even for the stiff case (υ=200).



Cell-Local IMEX-RK Temporal Evolution

At the heart of finite volume (FV) codes, like FLASH, is the reconstruction of time-averaging of states on either side of the interfaces between control volumes that make up the cells of the computational mesh. These states are given as the input to a Riemann problem, which yields fluxes between computational cells that are used to update the solution to the next time step. The accuracy of these reconstructions and local time evolutions are then vital to the overall accuracy of the code.



We take an ansatz solution on each cell as a modal expansion of each flow variable in terms of a set of orthogonal basis functions, in the style of ADER-CG/DG methods [2,5]. Similar expansions are carried out for the fluxes and source terms, and together with the governing PDE yields, through a Galerkin projection, a system of ODEs local to each computational cell. The novelty of this approach is that it allows us to use out-of-the-box ODE integrators to advance the system and produced time-averaged Riemann states. Specifically, we are interested in the class of mixed implicit explicit Runge-Kutta schemes that can treat stiff source terms implicitly for stability, while treating the advective fluxes explicitly for simplicity, without needing to operator split the treatment of the two terms.

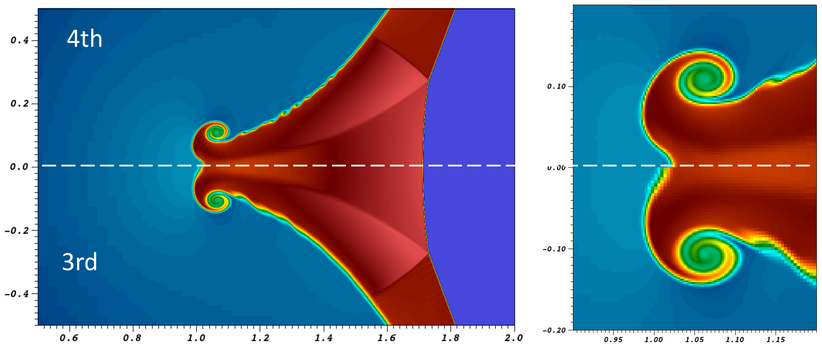
Accuracy Analysis

We test the accuracy of the proposed scheme on the isentropic vortex advection problem for the 2D Euler equations[4]. The problem consists of a nonlinear isentropic vortex that is advected through a periodic domain, returning to its initial configuration. Any discrepancy between the solution and the initial condition is due entirely to numerical errors. The table below presents L1 errors along with experimental orders of accuracy for the proposed schemes at 3rd and 4th order. Both schemes are able to meet their design orders of accuracy.

Δ	ADER-RK3		ADER-RK4	
	L_1	L_1 Order	L_1	L_1 Order
1/32	3.740508e-03	–	8.794432e-04	–
1/64	6.867190e-04	2.445442	6.317511e-05	3.799162
1/128	1.028993e-04	2.738487	2.164219e-06	4.867438
1/256	1.418378e-05	2.858919	9.477663e-08	4.513171

Shock Refraction

Lastly we test the proposed scheme on the shock refraction problem of Samtaney[3]. The problem stresses a numerical scheme's ability to adequately capture shocks and discontinuity while also resolving smooth features in the flow. This test involves a Mach 2 shock entering a reflecting tunnel incident on a contact discontinuity whose interface is oblique to the shock normal. The shock refracts through the contact discontinuity creating a vortex sheet that is unstable to the Kelving-Helmholtz instability. Small scale roll-ups along the shocked interface are seeded by numerical errors, and their presence while non-physical is indicative of low numerical diffusion in the scheme. Of more interest is the larger vortex roll up near the lower wall, whose winding is easily diffused by numerical diffusion in the scheme.



In the above figure we present solutions to the shock refraction problem using the propsed 3rd and 4th order ADER-RK methods on an adaptive grid with 7 levels of refinement with an equivalent resolution of 1024x512. Both solutions resolve the main features of the flow. The 4th order method is better able to maintain the interface over a smaller number of cells, including in the main vortex roll up, demonstrating the utility of higher-order methods, even on problems with shocks.

Future Work

- Extension to MHD on a staggered mesh to evolve face B fields (see E. Hansen's poster for more on staggered mesh MHD)
- Incorporate complex source terms, such as ion-electron heat exchange or chemically reactive flows
- Improve nonlinear solvers optimized for the local implicit solves

References

[1] L. Pareschi and G. Russo, "Implicit–Explicit Runge–Kutta Schemes and Applications to Hyperbolic Systems with Relaxation," *J Sci Comput*, vol. 25, no. 1, pp. 129–155, Oct. 2005, doi: 10.1007/s10915-004-4636-4.

[2] M. Dumbser and M. Käser, "Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems," *Journal of Computational Physics*, vol. 221, no. 2, pp. 693–723, Feb. 2007, doi: 10.1016/j.jcp.2006.06.043.

[3] R. Samtaney, "Computational Methods for Self-similar Solutions of the Compressible Euler Equations," *Journal of Computational Physics*, vol. 132, no. 2, pp. 327–345, Apr. 1997, doi: 10.1006/jcph.1996.5639.

[4] S. C. H. Spiegel, "A Survey of the Isentropic Euler Vortex Problem Using High-Order Methods," presented at the 22nd Computational Fluid Dynamics Conference, Dallas, TX, United States, Jun. 2015, Accessed: Nov. 04, 2019. [Online]. Available: https://ntrs.nasa.gov/search.jsp?R=20150018403.

[5] D. S. Balsara, T. Rumpf, M. Dumbser, and C.-D. Munz, "Efficient, high accuracy ADER-WENO schemes for hydrodynamics and divergence-free magnetohydrodynamics," *Journal of Computational Physics*, vol. 228, no. 7, pp. 2480–2516, Apr. 2009, doi: 10.1016/j.jcp.2008.12.003.

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